$\lambda-$STATISTICAL CONVERGENT FUNCTION SEQUENCES IN INTUITIONISTIC FUZZY NORMED SPACES

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1. Introduction

Fuzzy logic was introduced by Zadeh [1] in 1965. Since then, the importance of fuzzy logic has come increasingly to the present.There are many applications of fuzzy logic in the field of science and engineering, e.g. population dynamics [2], chaos control [3, 4], computer programming [5], nonlinear dynamical systems [6], etc. The concept of intuitionistic fuzzy set, as a generalization of fuzzy logic, was introduced by Atanassov [7] in 1986.

Quite recently Park [8] has introduced the concept of intuitionistic fuzzy metric space and in [9] , Saadati and Park studied the notion of intuitionistic fuzzy normed space. Intuitionistic fuzzy analogues of many concept in classical analysis was studied by many authors [10],[11],[13],[14],[19] etc.

The concept of statistical convergence was introduced by Fast [15]. Mursaleen defined $\lambda-$statistical convergence in[12]. Also the concept of statistical convergence was studied in intuitionistic fuzzy normed space in [16]..Quite recently, Karakaya et al. [22] defined and studied statistical convergence of function sequences in intuitionistic fuzzy normed spaces. Mohiuddine and Lohani defined and studied λ-statistical convergence in intuitionistic fuzzy normed spaces [17].

In this paper, we shall study concept λ-statistical convergence for function sequences and investigate some basic properties related to the concept in intuitionistic fuzzy normed space.

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Definition 1. [18] *A binary operation* $*:[0,1]\times[0,1]\to[0,1]$ *is said to be a continuous t-norm if it satisfies the following conditions:*
(i) $*$ *is associative and commutative ,*
(ii) $*$ *is continuous,*
(iii) $a*1=a$ *for all* $a\in[0,1]$ *,*
(iv) $a*c\le b*d$ *whenever* $a\le b$ *and* $c\le d$ *for each* $a,b,c,d\in[0,1]$

For example, $a*b=a.b$ is a continuous t-norm.

Definition 2. [18] *A binary operation* $\diamond:[0,1]\times[0,1]\to[0,1]$ *is said to be a continuous t-conorm if it satisfies the following conditions:*
(i) $\diamond$ *is associative and commutative ,*
(ii) $\diamond$ *is continuous,*
(iii) $a\diamond 0=a$ *for all* $a\in[0,1]$ *,*
(iv) $a\diamond c\le b\diamond d$ *whenever* $a\le b$ *and* $c\le d$ *for each* $a,b,c,d\in[0,1]$

For example, $a \diamond b = \min\{a+b, 1\}$ is a continuous t-norm.

Definition 3. [9] *Let $*$ be a continuous t-norm, $\diamond$ be a continuous t-conorm and X be a linear space over the field IF($\mathbb{R}$ or $\mathbb{C}$) . If μ and ν are fuzzy sets on $X \times (0,\infty)$ satisfying the following conditions, the five-tuple $(X,\mu,\nu,*,\diamond)$ is said to be an intuitionistic fuzzy normed space and (μ,ν) is called an intuitionistic fuzzy norm. For every $x,y \in X$ and $s,t>0$,*
(i) $\mu(x,t) + \nu(x,t) \leq 1$,
(ii) $\mu(x,t) > 0$,
(iii) $\mu(x,t) = 1 \Longleftrightarrow x = 0$,
(iv) $\mu(ax,t) = \mu\left(x, \frac{t}{|a|}\right)$ *for each* $a \neq 0$,
(v) $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,
(vi) $\mu(x,.) : (0,\infty) \to [0,1]$ *is continuous.*
(vii) $\lim\limits_{t\to\infty} \mu(x,t) = 1$ *and* $\lim\limits_{t\to 0} \mu(x,t) = 0$,
(viii) $\nu(x,t) < 1$,
(ix) $\nu(x,t) = 0 \Longleftrightarrow x = 0$,
(x) $\nu(ax,t) = \nu\left(x, \frac{t}{|a|}\right)$ *for each* $a \neq 0$,
(xi) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y,t+s)$,
(xii)$\nu(x,.) : (0,\infty) \to [0,1]$*is continuous.*
(xiii) $\lim\limits_{t\to\infty} \nu(x,t) = 1$ *and* $\lim\limits_{t\to 0} \nu(x,t) = 0$,

For the intuitionistic fuzzy normed space $(X,\mu,\nu,*,\diamond)$, as given in Dinda and Samanta [19]*, we further assume that $\mu,\nu,*,,\diamond$ satisfy the following axiom:*

(xiv) $\left.\begin{array}{c} a \diamond a = a \\ a * a = a \end{array}\right\}$ *for all* $a \in [0,1]$.

Definition 4. [9] *Let $(X,\mu,\nu,*,\diamond)$ be intuitionistic fuzzy normed space and (x_k) be sequence in X. (x_k) is said to be convergent to $L \in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if for every $\varepsilon > 0$ and $t > 0$, there exists a positive integer k_0 such that $\mu(x_k - L, t) > 1 - \varepsilon$ and $\nu(x_k - L, t) < \varepsilon$ whenever $k > k_0$. In this case we write $(\mu,\nu) - \lim x_k = L$ as $k \to \infty$.*

Definition 5. [9] *Let $(X,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy normed space. For t > 0, we define open ball $B(x,r,t)$ with center $x \in X$ and radius $0 < r < 1$, as*

$$B(x,r,t) = \{y \in X : \mu(x-y,t) > 1-r, \nu(x-y,t) < r\}$$

Definition 6. [19] *Let $(X,\mu,\nu,*,\diamond)$ and $(Y,\mu',\nu',*,\diamond)$ be two intuitionistic fuzzy normad linear space over the same field IF. A mapping f from $(X,\mu,\nu,*,\diamond)$ to $(Y,\mu',\nu',*,\diamond)$ is said to be intuitionistic fuzzy continuous at $x_0 \in X$, if for any given $\varepsilon > 0, a \in (0,1), \exists\delta = \delta(a,\varepsilon), \beta = \beta(a,\varepsilon) \in (0,1)$ such that for all $x \in X$,*

$$\mu(x - x_0, \delta) > 1 - \beta \Longrightarrow \mu\prime(f(x) - f(x_0), \varepsilon) > 1 - a$$

, and $\nu(x - x_0, \delta) < \beta \Longrightarrow \nu\prime(f(x) - f(x_0), \varepsilon) < a$.

Definition 7. [19] *Let $f_k : (X,\mu,\nu,*,\diamond) \to (Y,\mu',\nu',*,\diamond)$ be a sequence of functions. The sequence (f_k) is said to be pointwise intuitionistic fuzzy convergent on X to a function f with respect to (μ,ν) if for each $x \in X$, the sequence $(f_k(x))$ is convergent to $f(x)$ with respect to $(\mu\prime,\nu\prime)$.*

Definition 8. [19] *Let $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be a sequence of functions. The sequence (f_k) is said to be uniformly intuitionistic fuzzy convergent on X to a function f with respect to (μ, ν), if given $0 < r < 1, t > 0$, there exist a positive integer $k_0 = k_0(r, t)$ such that $\forall x \in X$ and $\forall k > k_0$,*

$$\mu\prime(f_k(x) - f(x), t) > 1 - r,\ \ \nu\prime(f_k(x) - f(x), t) < r.$$

Now, we recall the notion of the statistical convergence of sequences in intuitionistic fuzzy normed spaces.

Definition 9. [20] *Let $K \subset \mathbb{N}$ and $K_n = \{k \in K : k \leq n\}$.Then the natural density is defined by $\delta(K) = \lim\limits_{n\to\infty} \frac{|K_n|}{n}$, where $|K_n|$ denotes the cardinality of K_n.*

Definition 10. [21] *A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$, the set $N(\varepsilon)$ has asymptotic density zero, where*

$$N(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}.$$

This case is stated by $st - \lim x = L$.

Definition 11. *Let A be subset of $\mathbb{N}$. If a property P(k) holds for all $k \in A$ with $\delta(A) = 1$, we say that P holds for almost all k, that is a.a.k*

Definition 12. [16] *Let $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space. Then, a sequence (x_k) is said to be statistically convergent to $L \in X$ with respect to intuitionistic fuzzy norm (μ, ν) provided that for every $\varepsilon > 0$ and $t > 0$,*

$$\delta(\{k \in \mathbb{N} : \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - L, t) \geq \varepsilon\}) = 0$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - L, t) \geq \varepsilon\}| = 0$$

This case is stated by $st_{\mu,\nu} - \lim(x_k) = L$.

Definition 13. [22]*Let $(X, \mu, \nu, *, \diamond)$ and $(Y, \mu', \nu', *, \diamond)$ be two intuitionistic fuzzy normed linear spaces over the same field IF and $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be a sequence of functions. If for each $x \in X$ and $\forall\varepsilon > 0, t > 0$,*

$$\delta(\{k \in \mathbb{N} : \mu'(f_k(x) - f(x), t) \leq 1 - \varepsilon \text{ or } \nu'(f_k(x) - f(x), t) \geq \varepsilon\}) = 0,$$

then we say hat the sequence (f_k) is pointwise statistically convergent to f with respect to intuitionistic fuzzy norm (μ, ν) and we write it $st_{\mu,\nu} - f_k \to f$.

i.e., for each $x \in X$, $\mu'(f_k(x) - f(x), t) > 1 - \varepsilon$ and $\nu'(f_k(x) - f(x), t) < \varepsilon$ a.a.k.

Definition 14. [22]*Let $(X, \mu, \nu, *, \diamond)$ and $(Y, \mu', \nu', *, \diamond)$ be two intuitionistic fuzzy normed linear space over the same field IF and $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be a sequence of functions. If for every $x \in X$ and $\forall\varepsilon > 0, t > 0$,*

$$\delta(\{k \in \mathbb{N} : \mu'(f_k(x) - f(x), t) \leq 1 - \varepsilon \text{ or } \nu'(f_k(x) - f(x), t) \geq \varepsilon\}) = 0,$$

we say that the sequence (f_k) is uniformly statistically convergent with respect to f intuitionistic fuzzy norm (μ, ν) and we write it $st_{\mu,\nu} - f_k \rightrightarrows f$.

Definition 15. [16]*Let $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space. A sequence (x_k) is said to be statistically Cauchy with respect to intuitionistic fuzzy norm (μ, ν) provided that for every $\varepsilon > 0$ and $t > 0$, there exists a number $m \in \mathbb{N}$ satisfying*

$$\delta\left(\left\{k \in \mathbb{N} : \mu\left(x_k - x_m, t\right) \leq 1 - \varepsilon \text{ or } \nu\left(x_k - x_m, t\right) \geq \varepsilon\right\}\right) = 0.$$

Definition 16. [12] *Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that*

$$\lambda_{n+1} \leq \lambda_n, \qquad \lambda_1 = 0.$$

Let $K \subset \mathbb{N}$. The number

$$\delta_\lambda(K) = \lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : k \in K\right\}\right|$$

is said to be $\lambda-$density of K, where $I_n = [n - \lambda_n + 1, n]$.
If $\lambda_n = n$ for every n then $\lambda-$density is reduced to asymtotic density.

Definition 17. [12] *A sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent to the number L if for every $\varepsilon > 0$, the set $N(\varepsilon)$ has $\lambda-$density zero, where*

$$N(\varepsilon) = \left\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\right\}.$$

This case is stated by $st_\lambda - \lim x = L$.

Definition 18. [17] *Let $(X, \mu, \nu, *, \diamond)$ be an intuitionistic fuzzy normed space. Then, a sequence (x_k) is said to be $\lambda-$ statistically convergent to $L \in X$ with respect to intuitionistic fuzzy norm (μ, ν) provided that for every $\varepsilon > 0$ and $t > 0$,*

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mu\left(x_k - L, t\right) \leq 1 - \varepsilon \text{ or } \nu\left(x_k - L, t\right) \geq \varepsilon\right\}\right) = 0$$

or equivalently

$$\lim_{n\to\infty} \delta_\lambda\left(\left\{k \in \mathbb{N} : \mu\left(x_k - L, t\right) > 1 - \varepsilon \text{ and } \nu\left(x_k - L, t\right) < \varepsilon\right\}\right) = 1$$

This case is stated by $st^\lambda_{\mu,\nu} - \lim x = L$.

2. $\lambda-$Statistical Convergence of Sequence of Functions in Intuitionistic Fuzzy Normed Spaces

In this section, we define pointwise $\lambda-$statistical and uniformly $\lambda-$statistical convergent sequences of functions in intuitionistic fuzzy normed spaces. Also, we give the $\lambda-$statistical analog of the Cauchy convergence criterion for pointwise and uniformly $\lambda-$statistical convergent in intuitionistic fuzzy normed space.We investigate relationship of these concepts with continuity.

2.1. Pointwise $\lambda-$Statistical Convergence on intuitionistic fuzzy normed spaces.

Definition 19. *Let $(X, \mu, \nu, *, \diamond)$ and $(Y, \mu', \nu', *, \diamond)$ be two intuitionistic fuzzy normed linear spaces over the same field IF and $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ be a sequence of functions. If for each $x \in X$ and $\forall\varepsilon > 0, t > 0$,*

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x), t\right) \leq 1 - \varepsilon \text{ or } \nu'\left(f_k(x) - f(x), t\right) \geq \varepsilon\right\}\right) = 0,$$

or equivalently

$$\delta_\lambda\left(\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x), t\right) > 1 - \varepsilon \text{ and } \nu'\left(f_k(x) - f(x), t\right) < \varepsilon\right\}\right) = 1$$

then we say hat the sequence (f_k) *is pointwise* $\lambda-$*statistically convergent with respect to intuitionistic fuzzy norm* (μ, ν) *and we write it* $st^{\lambda}_{\mu,\nu} - f_k \to f$.

Remark 1.
Let $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ *be a sequence of functions. If* $\lambda_n = n$ *for every* n *,since* $\lambda -$ *density is reduced to asymtotic density, then* (f_k) *is pointwise statistically convergent on* X *with respect to* (μ, ν) *i.e.* $st_{\mu,\nu} - f_k \to f$.

Lemma 1. *Let* $f_k : (X, \mu, \nu, *, \diamond) \to (Y, \mu', \nu', *, \diamond)$ *be a sequence of functions.Then for every* $\varepsilon > 0$ *and* $t > 0$, *the following statements are equivalent:*

(i)$st^{\lambda}_{\mu,\nu} - f_k \to f$.
(ii)For each $x \in X$,

$$\delta_\lambda \left\{k \in \mathbb{N} : \mu' \left(f_k\left(x\right) - f\left(x\right), t\right) \leq 1 - \varepsilon \right\} = \delta_\lambda \left\{k \in \mathbb{N} :\ \nu' \left(f_k\left(x\right) - f\left(x\right), t\right) \geq \varepsilon\right\} = 0$$

(iii)$\delta_\lambda \left\{k \in \mathbb{N} : \mu' \left(f_k\left(x\right) - f\left(x\right), t\right) > 1 - \varepsilon \text{ and } \nu' \left(f_k\left(x\right) - f\left(x\right), t\right) < \varepsilon\right\} = 1$
(iv)For each $x \in X$,

$$\delta_\lambda \left\{k \in \mathbb{N} : \mu' \left(f_k\left(x\right) - f\left(x\right), t\right) > 1 - \varepsilon \right\} = \delta_\lambda \left\{k \in \mathbb{N} :\ \nu' \left(f_k\left(x\right) - f\left(x\right), t\right) < \varepsilon\right\} = 1$$

(v)For each $x \in X$,

$$st_\lambda - lim\mu'(f_k\left(x\right) - f\left(x\right), t) = 1 \text{ and } st_\lambda - lim\nu'(f_k\left(x\right) - f\left(x\right), t) = 0.$$

Example 1. *Let* $(\mathbb{R}, |\cdot|)$ *denote the space of real numbers with the usual norm, and let* $a * b = a.b$ *and* $a \diamond b = \min\{a + b, 1\}$ *for* $a, b \in [0, 1]$. *For all* $x \in \mathbb{R}$ *and every* $t > 0$, *consider*

$$\mu\left(x, t\right) = \frac{t}{t + |x|} \text{ and } \nu\left(x, t\right) = \frac{|x|}{t + |x|}$$

In this case $(\mathbb{R}, \mu, \nu, *, \diamond)$ *is intuitionistic fuzzy normed space. Let* $f_k : [0, 1] \to \mathbb{R}$ *be a sequence of functions whose terms are given by*

$$f_k(x) = \left\{ \begin{array}{cc} x^k + 1, & \text{for } 0 \leq x < \tfrac{1}{2}, \text{ if } n - \sqrt{\lambda_n} + 1 \leq k \leq n \\ 0\ , & \text{for } 0 \leq x < \tfrac{1}{2}, \text{ otherwise} \\ x^k + \tfrac{1}{2}, & \text{for } \tfrac{1}{2} \leq x < 1, \text{ if if } n - \sqrt{\lambda_n} + 1 \leq k \leq n \\ 1, & \text{for } \tfrac{1}{2} \leq x < 1, \text{otherwise} \\ 2, & \text{for } x = 1 \end{array} \right. .$$

(f_k) *is pointwise* $\lambda -$ *statistical convergent on* $[0, 1]$ *with respect to intuitionistic fuzzy norm* (μ, ν). *Because, for* $0 \leq x < \frac{1}{2}$, *since*

$$K\left(\varepsilon, t\right) = \left\{k \in \mathbb{N} :\ \mu\left(f_k(x) - f(x), t\right) \leq 1 - \varepsilon \text{ or } \nu\left(f_k(x) - f(x), t\right) \geq \varepsilon\right\},$$

hence

$$\begin{aligned} K\left(\varepsilon, t\right) &= \left\{k \in I_n : \frac{t}{t + |f_k(x) - 0|} \leq 1 - \varepsilon \text{ or } \frac{|f_k(x) - 0|}{t + |f_k(x) - 0|} \geq \varepsilon\right\} \\ &= \left\{k \in I_n : |f_k(x)| \geq \frac{\varepsilon t}{1 - \varepsilon}\right\} \\ &= \left\{k \in I_n : f_k(x) = x^k + 1\right\} \end{aligned}$$

and

$$|K\left(\varepsilon, t\right)| \leq \sqrt{\lambda_n}$$

Thus, for $0 \leq x < \frac{1}{2}$, *since*

$$\delta_\lambda\left(K\left(\varepsilon,t\right)\right)=\lim_{n\to\infty}\frac{\left|K\left(\varepsilon,t\right)\right|}{\lambda_n}=\lim_{n\to\infty}\frac{\sqrt{\lambda_n}}{\lambda_n}=0$$

f_k *is* $\lambda-$*statistical convergent to* 0 *with respect to intuitionistic fuzzy norm* (μ,ν).

For $\frac{1}{2} \leq x < 1$,

$$\begin{aligned} K'\left(\varepsilon,t\right) &= \left\{k\in I_n : \frac{t}{t+|f_k(x)-1|}\leq 1-\varepsilon \text{ or } \frac{|f_k(x)-1|}{t+|f_k(x)-1|}\geq\varepsilon\right\} \\ &= \left\{k\in I_n : |f_k(x)-1|\geq\frac{\varepsilon t}{1-\varepsilon}\right\} \\ &= \left\{k\in I_n : f_k(x)=x^k+\frac{1}{2}\right\} \end{aligned}$$

and

$$\left|K'\left(\varepsilon,t\right)\right|\leq\sqrt{\lambda_n}$$

Thus, for $0 \leq x < \frac{1}{2}$, f_k is $\lambda-$ *statistical* convergent to 1with respect to intuitionistic fuzzy norm (μ,ν).

For $x=1$, it can be seen easly that

$$\begin{aligned} K''\left(\varepsilon,t\right) &= \left\{k\in\mathbb{N} : \frac{t}{t+|f_k(x)-2|}\leq 1-\varepsilon \text{ or } \frac{|f_k(x)-2|}{t+|f_k(x)-2|}\geq\varepsilon\right\} \\ &= \left\{n\in\mathbb{N} : 0\geq\frac{\varepsilon t}{1-\varepsilon}\right\} \\ &= \oslash \end{aligned}$$

and

$$\left|K''\left(\varepsilon,t\right)\right|=0$$

and

$$\delta_\lambda\left(K''\left(\varepsilon,t\right)\right)=\lim_{n\to\infty}\frac{\left|K''\left(\varepsilon,t\right)\right|}{\lambda_n}=\lim_{n\to\infty}\frac{0}{\lambda_n}=0.$$

Thus for $x=1$, f_k is $\lambda-$*statistical* convergent to 2 with respect to intuitionistic fuzzy norm (μ,ν)

Theorem 1. *Let* $(X,\mu,\nu,*,\diamond)$ *be an intuitionistic fuzzy normed space and* $f_k : (X,\mu,\nu,*,\diamond)\to(Y,\mu',\nu',*,\diamond)$ *be a sequence of functions.If sequence* (f_k) *is pointwise intuitionistic fuzzy convergent on* X *to a function* f *with respect to* (μ,ν), *then* (f_k) *is pointwise* $\lambda-$*statistical convergent with respect to intuitionistic fuzzy norm* (μ,ν).

Proof. Let $\forall k\in\mathbb{N}$ and (f_k) be pointwise intuitionistic fuzzy convergent in X. In this case the sequence $(f_k(x))$ is convergent with respect to (μ',ν') for each $x\in X$. Then for every $\varepsilon>0$ and $t>0$, there is number $k_0\in\mathbb{N}$ such that

$$\mu'\left(f_k\left(x\right)-f\left(x\right),t\right)>1-\varepsilon \text{ and } \nu'\left(f_k\left(x\right)-f\left(x\right),t\right)<\varepsilon$$

for all $\forall k\geq k_0$ and for each $x\in X$.Hence for each $x\in X$ the set

$$\left\{k\in\mathbb{N} : \mu'\left(f_k\left(x\right)-f\left(x\right),t\right)\leq 1-\varepsilon \text{ or } \nu'\left(f_k\left(x\right)-f\left(x\right),t\right)\geq\varepsilon\right\}$$

has finite numbers of terms. Since finite subset of $\mathbb{N}$ has $\lambda-$density 0 and hence

$$\delta_\lambda\left(\{k\in\mathbb{N}:\mu'\left(f_k\left(x\right)-f\left(x\right),t\right)\leq 1-\varepsilon \text{ or } \nu'\left(f_k\left(x\right)-f\left(x\right),t\right)\geq\varepsilon\}\right)=0.$$

That is, $st^\lambda_{\mu,\nu}-f_k\to f$. $\square$

Theorem 2. *Let (f_k) and (g_k) be two sequences of functions from intuitionistic fuzzy normed space $(X,\mu,\nu,*,\diamond)$ to $(Y,\mu',\nu',*,\diamond)$. If $st^\lambda_{\mu,\nu}-f_k\to f$ and $st^\lambda_{\mu,\nu}-g_k\to g$, then $st^\lambda_{\mu,\nu}-(\alpha f_k+\beta g_k)\to\alpha f+\beta g$ where $\alpha,\beta\in IF(\mathbb{R}$ or $\mathbb{C})$.*

Proof. The proof is clear for $\alpha=0$ and $\beta=0$.Now let $\alpha\neq 0$ and $\beta\neq 0$. Since $st^\lambda_{\mu,\nu}-f_k\to f$ and $st^\lambda_{\mu,\nu}-g_k\to g$, for each $x\in X$ if we define

$$A_1=\left\{k\in\mathbb{N}:\mu'\left(f_k\left(x\right)-f\left(x\right),\frac{t}{2\left|\alpha\right|}\right)\leq 1-\varepsilon \text{ or } \nu'\left(f_k\left(x\right)-f\left(x\right),\frac{t}{2\left|\alpha\right|}\right)\geq\varepsilon\right\}$$

and

$$A_2=\left\{k\in\mathbb{N}:\ \mu'\left(g_k\left(x\right)-g\left(x\right),\frac{t}{2\left|\beta\right|}\right)\leq 1-\varepsilon \text{ or } \nu'\left(g_k\left(x\right)-g\left(x\right),\frac{t}{2\left|\beta\right|}\right)\geq\varepsilon\right\}$$

then

$$\delta_\lambda\left(A_1\right)\ =0\quad\text{and } \delta_\lambda\left(A_2\right)\ =0\ .$$

Since $\delta_\lambda\left(A_1\right)=0$ and $\delta_\lambda\left(A_2\right)=0$, if we state A by $(A_1\cup A_2)$ then

$$\delta_\lambda\left(A\right)=0.$$

Hence $A_1\cup A_2\neq\mathbb{N}$ and there exists $\exists k_0\in\mathbb{N}$ such that

$$\begin{aligned}\mu'\left(f_{k_0}\left(x\right)-f\left(x\right),\frac{t}{2\left|\alpha\right|}\right) &> 1-\varepsilon,\nu'\left(f_{k_0}\left(x\right)-f\left(x\right),\frac{t}{2\left|\alpha\right|}\right)<\varepsilon,\\ \mu'\left(g_{k_0}\left(x\right)-g\left(x\right),\frac{t}{2\left|\beta\right|}\right) &> 1-\varepsilon \text{ and } \nu'\left(g_{k_0}\left(x\right)-g\left(x\right),\frac{t}{2\left|\beta\right|}\right)<\varepsilon\end{aligned}$$

Let

$$\begin{aligned}B &= \{k\in\mathbb{N}:\mu'\left(\left(\alpha f_k+\beta g_k\right)\left(x\right)-\left(\alpha f(x)+\beta g(x)\right),t\right)>1-\varepsilon \text{ and}\\ &\qquad \nu'\left(\left(\alpha f_k+\beta g_k\right)\left(x\right)-\left(\alpha f(x)+\beta g(x)\right),t\right)<\varepsilon\}.\end{aligned}$$

We shall show that for each $x\in X$

$$A^c\subset B$$

Let $k_0\in A^c$.In this case

$$\mu'\left(f_{k_0}\left(x\right)-f\left(x\right),\frac{t}{2\left|\alpha\right|}\right)>1-\varepsilon,\nu'\left(f_{k_0}\left(x\right)-f\left(x\right),\frac{t}{2\left|\alpha\right|}\right)<\varepsilon,$$

and

$$\mu'\left(g_{k_0}\left(x\right)-g\left(x\right),\frac{t}{2\left|\beta\right|}\right)>1-\varepsilon \text{ and } \nu'\left(g_{k_0}\left(x\right)-g\left(x\right),\frac{t}{2\left|\beta\right|}\right)<\varepsilon.$$

Using those above, we have

$$\begin{aligned}\mu'\left(\left(\alpha f_{k_0}+\beta g_{k_0}\right)(x)-\left(\alpha f(x)+\beta g(x)\right),t\right) &\geq \mu'\left(\alpha f_{k_0}(x)-\alpha f(x),\frac{t}{2}\right)*\mu'\left(\beta g_{k_0}(x)-\beta g(x),\frac{t}{2}\right)\\ &= \mu'\left(f_{k_0}(x)-f(x),\frac{t}{2|\alpha|}\right)*\mu'\left(g_{k_0}(x)-g(x),\frac{t}{2|\beta|}\right)\\ &> (1-\varepsilon)*(1-\varepsilon)\\ &= (1-\varepsilon)\end{aligned}$$

and

$$\begin{aligned}\nu'\left(\left(\alpha f_{k_0}+\beta g_{k_0}\right)(x)-\left(\alpha f(x)+\beta g(x)\right),t\right) &\leq \nu'\left(\alpha f_{k_0}(x)-\alpha f(x),\frac{t}{2}\right)*\nu'\left(\beta g_{k_0}(x)-\beta g(x),\frac{t}{2}\right)\\ &= \nu'\left(f_{k_0}(x)-f(x),\frac{t}{2|\alpha|}\right)*\nu'\left(g_{k_0}(x)-g(x),\frac{t}{2|\beta|}\right)\\ &< \varepsilon\diamond\varepsilon\\ &= \varepsilon\end{aligned}$$

This implies that

$$A^c\subset B.$$

Since $B^c\subset A$ and $\delta_\lambda(A)=0$, hence

$$\delta_\lambda(B^c)=0$$

That is

$$\delta_\lambda\left(\left\{k\in\mathbb{N}:\mu'\left(\left(\alpha f_k+\beta g_k\right)(x)-\left(\alpha f(x)+\beta g(x)\right),t\right)\leq 1-\varepsilon \text{ and } \nu'\left(\left(\alpha f_k+\beta g_k\right)(x)-\left(\alpha f(x)+\beta g(x)\right),t\right)\geq\varepsilon\right.$$

$$st^\lambda_{\mu,\nu}-\left(\alpha f_k+\beta g_k\right)\to\alpha f+\beta g$$

□

Definition 20. *Let $f_k:(X,\mu,\nu,*,\diamond)\to(Y,\mu',\nu',*,\diamond)$ be a sequence of functions. The sequence (f_k) is a pointwise $\lambda-$statistical Cauchy sequence in intuitionistic fuzzy normed space provided that for every $\varepsilon>0$ and $t>0$ there exists a number $N=N(x,\varepsilon,t)$ such that*

$$\delta_\lambda\left(\left\{k\in\mathbb{N}:\mu'\left(f_k(x)-f_N(x),t\right)\leq 1-\varepsilon \text{ or } \nu'\left(f_k(x)-f_N(x),t\right)\geq\varepsilon \text{ for each } x\in X\right\}\right)=0.$$

Theorem 3. *Let $f_k:(X,\mu,\nu,*,\diamond)\to(Y,\mu',\nu',*,\diamond)$ be a sequence of functions. If (f_k) is a pointwise $\lambda-$statistical convergent sequence with respect to intuitionistic fuzzy norm (μ,ν), then (f_k) is a pointwise $\lambda-$statistical Cauchy sequence with respect to intuitionistic fuzzy norm (μ,ν).*

Proof. Suppose that $st^\lambda_{\mu,\nu}-f_k\to f$ and let $\varepsilon>0$, $t>0$. For a given $\varepsilon>0$, choose $s>0$ such that $(1-\varepsilon)*(1-\varepsilon)>1-s$ and $\varepsilon\diamond\varepsilon<s$. If we state respectively $A_x(\varepsilon,t)$ and $A^c_x(\varepsilon,t)$ by

$$\left\{k\in\mathbb{N}:\mu'\left(f_k(x)-f(x),\frac{t}{2}\right)\leq 1-\varepsilon \text{ or } \nu'\left(f_k(x)-f(x),\frac{t}{2}\right)\geq\varepsilon\right\},$$

$$\left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x), \frac{t}{2}\right) > 1 - \varepsilon \text{ and } \nu'\left(f_k(x) - f(x), \frac{t}{2}\right) < \varepsilon\right\}$$

for each $x \in X$ .Then, we have

$$\delta_\lambda\left(A_x(\varepsilon, t)\right) = 0$$

which implies that

$$\delta_\lambda\left(A_x^c(\varepsilon, t)\right) = 1$$

Let $N \in A_x^c(\varepsilon, t)$ .Then

$$\mu'\left(f_N(x) - f(x), \frac{t}{2}\right) > 1 - \varepsilon \text{ and } \nu'\left(f_N(x) - f(x), \frac{t}{2}\right) < \varepsilon$$

We want to show that there exists a number $N = N(x, \varepsilon, t)$ such that

$$\delta_\lambda\left(\{k \in \mathbb{N} : \mu'\left(f_k(x) - f_N(x), t\right) \leq 1 - s \text{ or } \nu'\left(f_k(x) - f_N(x), t\right) \geq s \text{ for each } x \in X\}\right) = 0.$$

Therefore, define for each $x \in X$,

$$B_x(\varepsilon, t) = \{k \in \mathbb{N} : \mu'\left(f_k(x) - f_N(x), t\right) \leq 1 - s \text{ or } \nu'\left(f_k(x) - f_N(x), t\right) \geq s\ \}.$$

We have to show that

$$B_x(\varepsilon, t) \subset A_x(\varepsilon, t).$$

Suppose that

$$B_x(\varepsilon, t) \nsubseteq A_x(\varepsilon, t).$$

In this case $B_x(\varepsilon, t)$ has at least one different element which $A_x(\varepsilon, t)$ doesn't has.
Let $k \in B_x(\varepsilon, t) \diagdown A_x(\varepsilon, t)$. Then we have

$$\mu'\left(f_k(x) - f_N(x), t\right) \leq 1 - s \text{ and } \mu'\left(f_k(x) - f(x), \frac{t}{2}\right) > 1 - \varepsilon,$$

in particularly $\mu'\left(f_N(x) - f(x), \frac{t}{2}\right) > 1 - \varepsilon$. In this case

$$\begin{aligned} 1 - s &\geq \mu'\left(f_k(x) - f_N(x), t\right) \geq \mu'\left(f_k(x) - f(x), \frac{t}{2}\right) * \mu'\left(f_N(x) - f(x), \frac{t}{2}\right) \\ &\geq (1 - \varepsilon) * (1 - \varepsilon) > 1 - s, \end{aligned}$$

which is not possible. On the other hand

$$\nu'\left(f_k(x) - f_N(x), t\right) \geq s \text{ and } \nu'\left(f_k(x) - f(x), t\right) < \varepsilon,$$

in particularly $\nu'\left(f_N(x) - f(x), t\right) < \varepsilon$.In this case

$$s \leq \nu'\left(f_k(x) - f_N(x), t\right) \leq \nu'\left(f_k(x) - f(x), \frac{t}{2}\right) \diamond \nu'\left(f_N(x) - f(x), \frac{t}{2}\right)$$

$$< \varepsilon \diamond \varepsilon < s$$

which is not possible. Hence $B_x(\varepsilon, t) \subset A_x(\varepsilon, t)$. Therefore, by $\delta_\lambda\left(A_x(\varepsilon, t)\right) = 0$, $\delta_\lambda\left(B_x(\varepsilon, t)\right) = 0$. That is; $(f_k)$is a pointwise $\lambda-$statistical Cauchy sequence with respect to intuitionistic fuzzy norm (μ, ν). $\square$

2.2. Uniformly $\lambda-$Statistical Convergence on intuitionistic fuzzy normed spaces.

Definition 21. *Let $(X,\mu,\nu,*,\diamond)$ and $(Y,\mu',\nu',*,\diamond)$ be two intuitionistic fuzzy normed linear spaces over the same field IF and $f_k:(X,\mu,\nu,*,\diamond)\to(Y,\mu',\nu',*,\diamond)$ be a sequence of functions. If for every $x\in X$ and $\forall\varepsilon>0, t>0$,*

$$\delta_\lambda\left(\{k\in\mathbb{N}:\mu'\left(f_k\left(x\right)-f\left(x\right),t\right)\leq 1-\varepsilon \text{ or } \nu'\left(f_k\left(x\right)-f\left(x\right),t\right)\geq\varepsilon\}\right)=0,$$

or equivalently

$$\delta_\lambda\left(\{k\in\mathbb{N}:\mu'\left(f_k\left(x\right)-f\left(x\right),t\right)>1-\varepsilon \text{ and } \nu'\left(f_k\left(x\right)-f\left(x\right),t\right)<\varepsilon\}\right)=1$$

then we say hat the sequence f_k is uniformly $\lambda-$statistical convergent with respect to intuitionistic fuzzy norm (μ,ν) and we write it $st^\lambda_{\mu,\nu}-f_k\rightrightarrows f$ and

Remark 2. *If $st^\lambda_{\mu,\nu}-f_k\rightrightarrows f$, then $st^\lambda_{\mu,\nu}-f_k\to f$.*

Lemma 2. *Let $f_k:(X,\mu,\nu,*,\diamond)\to(Y,\mu',\nu',*,\diamond)$ be a sequence of functions.Then for every $\varepsilon>0$ and $t>0$, the following statements are equivalent:*

(i)$st^\lambda_{\mu,\nu}-f_k\rightrightarrows f$.
(ii)For every $x\in X$,

$$\delta_\lambda\left\{k\in\mathbb{N}:\mu'\left(f_k\left(x\right)-f\left(x\right),t\right)\leq 1-\varepsilon\ \right\}=\delta_\lambda\left\{k\in\mathbb{N}:\ \nu'\left(f_k\left(x\right)-f\left(x\right),t\right)\geq\varepsilon\right\}=0$$

(iii)For every $x\in X$,

$$\delta_\lambda\left\{k\in\mathbb{N}:\mu'\left(f_k\left(x\right)-f\left(x\right),t\right)>1-\varepsilon \text{ and } \nu'\left(f_k\left(x\right)-f\left(x\right),t\right)<\varepsilon\right\}=1$$

(iv)For every $x\in X$,

$$\delta_\lambda\left\{k\in\mathbb{N}:\mu'\left(f_k\left(x\right)-f\left(x\right),t\right)>1-\varepsilon\ \right\}=\delta_\lambda\left\{k\in\mathbb{N}:\ \nu'\left(f_k\left(x\right)-f\left(x\right),t\right)<\varepsilon\right\}=1$$

(v)For every $x\in X$,

$$st_\lambda-lim\mu'(f_k\left(x\right)-f\left(x\right),t)=1 \text{ and } st_\lambda-lim\nu'(f_k\left(x\right)-f\left(x\right),t)=0.$$

Theorem 4. *Let $(X,\mu,\nu,*,\diamond)$ be an intuitionistic fuzzy normed space and $f_k:(X,\mu,\nu,*,\diamond)\to(Y,\mu',\nu',*,\diamond)$ be a sequence of functions.If sequence (f_k) is uniformly intuitionistic fuzzy convergent on X to a function f with respect to (μ,ν), then (f_k) is uniformly $\lambda-$statistical convergent with respect to intuitionistic fuzzy norm (μ,ν).*

Proof. Proof of this theorem is similar to proof of theorem that we have done previously for pointwise $\lambda-statistical$ convergent. □

Theorem 5. *Let (f_k) and (g_k) be two sequences of functions from intuitionistic fuzzy normed space $(X,\mu,\nu,*,\diamond)$ to$(Y,\mu',\nu',*,\diamond)$. If $st^\lambda_{\mu,\nu}-f_k\rightrightarrows f$ and $st^\lambda_{\mu,\nu}-g_k\rightrightarrows g$, then $st^\lambda_{\mu,\nu}-(\alpha f_k+\beta g_k)\rightrightarrows\alpha f+\beta g$ where $\alpha,\beta\in IF(\mathbb{R}$ or $\mathbb{C})$.*

Example 2. *Let $(\mathbb{R},|\cdot|)$ denote the space of real numbers with the usual norm, and let $a*b=a.b$ and $a\diamond b=\min\{a+b,1\}$ for $a,b\in[0,1]$. For all $x\in\mathbb{R}$ and every $t>0$, consider*

$$\mu\left(x,t\right)=\frac{t}{t+|x|} \text{ and } \nu\left(x,t\right)=\frac{|x|}{t+|x|}$$

Let $f_k:[0,1]\to\mathbb{R}$ be a sequence of functions whose terms are given by

$$f_k(x)=\begin{cases} x^k+1, & \text{if } n-\sqrt{\lambda_n}+1\leq k\leq n \\ 0\ , & \text{otherwise}\end{cases}.$$

since $K(\varepsilon,t)=\{k\in I_n:\ \mu(f_k(x)-f(x),t)\leq 1-\varepsilon\ \text{or}\ \nu(f_k(x)-f(x),t)\geq\varepsilon\}$ *,hence*

$$\begin{aligned} K(\varepsilon,t) &= \left\{k\in I_n:\frac{t}{t+|f_k(x)-0|}\leq 1-\varepsilon\ \text{or}\ \frac{|f_k(x)-0|}{t+|f_k(x)-0|}\geq\varepsilon\right\} \\ &= \left\{k\in I_n:|f_k(x)|\geq\frac{\varepsilon t}{1-\varepsilon}\right\} \\ &= \left\{k\in I_n:f_k(x)=x^k+1\right\} \end{aligned}$$

and

$$|K(\varepsilon,t)|\leq\sqrt{\lambda_n}$$

Thus, for $0\leq x\leq 1$*, since*

$$\delta_\lambda(K(\varepsilon,t))=\lim_{n\to\infty}\frac{|K(\varepsilon,t)|}{\lambda_n}=\lim_{n\to\infty}\frac{\sqrt{\lambda_n}}{\lambda_n}=0$$

f_k *is uniform* $\lambda-$ *statistical convergent to* 0 *with respect to intuitionistic fuzzy norm* (μ,ν)*.*

Definition 22. *Let* $f_k:(X,\mu,\nu,*,\diamond)\to(Y,\mu',\nu',*,\diamond)$ *be a sequence of functions.The sequence* (f_k) *is a uniformly* $\lambda-$*statistical Cauchy sequence in intuitionistic fuzzy normed space provided that for every* $\varepsilon>0$ *and* $t>0$ *there exists a number* $N=N(\varepsilon,t)$ *such that*

$$\delta_\lambda(\{k\in\mathbb{N}:\mu'(f_k(x)-f_N(x),t)\leq 1-\varepsilon\ \text{or}\ \nu'(f_k(x)-f_N(x),t)\geq\varepsilon\ \text{for every}\ x\in X\})=0.$$

Theorem 6. *Let* $f_k:(X,\mu,\nu,*,\diamond)\to(Y,\mu',\nu',*,\diamond)$ *be a sequence of functions.If* (f_k) *is a uniformly* $\lambda-$*statistically convergent sequence with respect to intuitionistic fuzzy norm* (μ,ν)*, then* (f_k)*is a uniformly* $\lambda-$*statistical Cauchy sequence with respect to intuitionistic fuzzy norm* (μ,ν)*.*

Proof. Proof of this theorem is similar to proof of theorem that we have done previously for pointwise $\lambda-$ *statistical* convergent □

Definition 23. *Let* $(X,\mu,\nu,*,\diamond)$ *and* $(Y,\mu',\nu',*,\diamond)$ *be two intuitionistic fuzzy normed space, and* F *a family of functions from* X *to* Y*. The family* F *is intuitionistic fuzzy equicontinuous at a point* $x_0\in X$ *if for every* $\varepsilon>0$ *and* $t>0$*, there exists a* $\delta>0$ *such that* $\mu'(f(x_0)-f(x),t)>1-\varepsilon$ *and* $\nu'(f(x_0)-f(x),t)<\varepsilon$ *for all* $f\in F$ *and all* x *such that*$\mu'(x_0-x,t)>1-\delta$ *and* $\nu'(x_0-x,t)<\delta$*. The family is intuitionistic fuzzy equicontinuous if it is equicontinuous at each point of* X*.(For continuity,* δ *may depend on* ε*,* x_0 *and* f*; for equicontinuity,* δ *must be independent of* f*)*

Theorem 7. *let* $(X,\mu,\nu,*,\diamond),(Y,\mu',\nu',*,\diamond)$ *be intuitionistic fuzzy normed space.Assume that* $st^\lambda_{\mu,\nu}-f_k\to f$ *on* X *where functions* $f_k:(X,\mu,\nu,*,\diamond)\to(Y,\mu',\nu',*,\diamond),k\in\mathbb{N}$*, are intuitionistic fuzzy equi-continuous on* X *and* $f:X\to Y$*. Then* f *is continuous on* X*.*

Proof. Let $x_0\in X$ be an arbitrary point. By the intuitionistic fuzzy equi-continuity of f_k's,for every $\varepsilon>0$ and $t>0$ there exist $\delta=\delta\left(x_0,\varepsilon,\frac{t}{3}\right)>0$ such that

$$\mu'\left(f_k(x_0)-f_k(x),\frac{t}{3}\right)>1-\varepsilon\ \text{and}\ \nu'\left(f_k(x_0)-f_k(x),\frac{t}{3}\right)<\varepsilon$$

for every $k \in \mathbb{N}$ and all x such that $\mu'(x_0 - x, t) > 1 - \delta$ and $\nu'(x_0 - x, t) < \delta$. Let $x \in B(x_0, \delta, t)$ be fixed.Since $st^{\lambda}_{\mu,\nu} - f_k \to f$ on X,for each $x \in X$, if we state respectively $A_x(\varepsilon, t)$ and $B_{\ x}(\varepsilon, t)$ by the sets

$$A_x(\varepsilon,t) = \left\{k \in \mathbb{N} : \mu'\left(f_k(x) - f(x), \frac{t}{3}\right) \leq 1 - \varepsilon \text{ or } \nu'\left(f_k(x) - f(x), \frac{t}{3}\right) \geq \varepsilon \text{ for each } x \in X\right\}$$

and

$$B_x(\varepsilon,t) = \left\{k \in \mathbb{N} : \mu'\left(f_k(x_0) - f(x_0), \frac{t}{3}\right) \leq 1 - \varepsilon \text{ or } \nu'\left(f_k(x_0) - f(x_0), \frac{t}{3}\right) \geq \varepsilon \text{ for each } x \in X\right\}$$

then, $\delta_\lambda(A_x(\varepsilon,t)) = 0$ and $\delta_\lambda(B_x(\varepsilon,t)) = 0$, hence$\delta_\lambda(A_x(\varepsilon,t) \cup B_x(\varepsilon,t)) = 0$ and $A_x(\varepsilon,t) \cup B_x(\varepsilon,t)$ is different fom $\mathbb{N}$. Thus, there exists $\exists m \in \mathbb{N}$ such that

$$\mu'\left(f_m(x) - f(x), \frac{t}{3}\right) > 1 - \varepsilon\ ,\ \nu'\left(f_m(x) - f(x), \frac{t}{3}\right) < \varepsilon$$

and

$$\mu'\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) > 1 - \varepsilon\ ,\ \nu'\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) < \varepsilon.$$

Now, we will show that f is intuitionistic fuzzy contunious at x_0.Since $st^{\lambda}_{\mu,\nu} - f_k \to f$ and for every $k \in \mathbb{N}$ f_k's are continuous, f_m is also continuous for $m \in \mathbb{N}$, we have

$$\begin{aligned}
\mu\prime(f(x) - f(x_0), t) &= \mu\prime(f(x) - f_m(x) + f_m(x) - f_m(x_0) + f_m(x_0) - f(x_0), t) \\
&\geq \mu\prime\left(f(x) - f_m(x), \frac{t}{3}\right) * \mu\prime\left(f_m(x) - f_k(x_0), \frac{t}{3}\right) * \mu\prime\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) \\
&> 1 - \varepsilon * 1 - \varepsilon * 1 - \varepsilon \\
&= 1 - \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\nu\prime(f(x) - f(x_0), t) &= \nu\prime(f(x) - f_m(x) + f_m(x) - f_m(x_0) + f_m(x_0) - f(x_0), t) \\
&\leq \nu\prime\left(f(x) - f_m(x), \frac{t}{3}\right) * \nu\prime\left(f_m(x) - f_m(x_0), \frac{t}{3}\right) * \nu\prime\left(f_m(x_0) - f(x_0), \frac{t}{3}\right) \\
&< \varepsilon \diamond \varepsilon \diamond \varepsilon \\
&= \varepsilon.
\end{aligned}$$

Thus, the proof is completed. □

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